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Unified Approaches to Well-Posedness with Some Applications

HUI YANG and JIAN YU
Department of Mathematics, Guizhou University, Guiyang, Guizhou, 550025, China (e-mail: hui-yang@163.com)

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Abstract. We present unified approaches to Hadamard and Tykhonov well-posedness. As applications, we deduce Tykhonov well-posedness for optimization problems, Nash equilibrium point problems and fixed point problems etc. Especially, by applying such approaches, we deal with the well-posedness as stated in (Lignola and Morgan (2000), Journal of Global Optimization 16, 57–67) in which Lignola and Morgan investigated directly and intensively Tykhonov types of well-posedness for optimization problems with constraints defined by variational inequalities, namely, generalized well-posedness and strong well-posedness. We give some sufficient conditions for Hadamard well-posedness of such problems and deduce relations between Hadamard type and Tykhonov type of well-posedness. Finally, as corollaries, we derive generalized well-posedness and strong well-posedness for these problems.

Key words: Fixed point, Generalized well-posedness, Hadamard well-posedness, Nash equilibrium, Optimization problem, Strong well-posedness, Tykhonov well-posedness

1. Introduction

Recently, well-posedness for various types of nonlinear problems has attracted attentions of many researchers. In 1993, Dontchev and Zolezzi's monograph [2] was published. Ref. [3] includes several surveys on well-posedness of vector optimization problems, Nash equilibria and variational calculus etc. In recent years, many researchers investigated intensively well-posedness for some specific problems. For instance, Revalski [4] gave a survey on various aspects on well-posedness of optimization problems; Lignola and Morgan [1] investigated well-posedness for optimization problems with variational inequalities constraints; Margiocco et al. [5, 6] discussed Tykhonov well-posedness for Nash equilibria. Well-posedness for vector optimization problems has been investigated by Loridan [7] and Huang [8] etc.

As is well-known, the notions of well-posedness can be divided into two groups, namely, Hadamard type and Tykhonov type. Generally speaking, to consider Tykhonov type of well-posedness for some problems, one introduces the notion of 'approximating sequence' for the solutions and requires some convergence of such sequences to a solution of the problem. As for

Hadamard type of well-posedness of a problem, we mean the continuous dependence of the solutions on the data of such problem related to some topology on the problem space. Almost all the literature deals with directly specific notions of well-posedness, especially Tykhonov types of well-posedness. While some researchers have investigated the relations between them for different problems (see [3, 9, 10] etc.), there is no general research to such relations. Aim of this paper is to present a unified approach to both Hadamard and Tykhonov well-posedness. Precisely, we propose a general notion of Tykhonov well-posedness, which is suitable for many nonlinear problems including optimization problem, fixed point problem, Nash equilibria etc. We will discuss generally the relations between Hadamard and Tykhonov well-posedness. As applications, we deduce some well-posedness theorems for optimization problems, Nash equilibrium point problems and fixed point problem etc. Especially, we investigate well-posedness for optimization problems with constraints defined by variational inequalities.

2. Unified Approaches to Well-Posedness

In this section, we give general notions of Tykhonov well-posedness and Hadamard well-posedness and investigate relations between them.

Let X and Y be two Hausdorff topological spaces and $F : Y \to 2^X$ be a set-valued map. R_+ denotes the set of all nonnegative real-numbers.

DEFINITION 2.1. Let $y \in Y$ and suppose that there is $\varphi : X \to R_+$ such that $\varphi(x) = 0$ if and only if $x \in F(y)$.

(i) If for any $x_n \in X$, $\varphi(x_n) \to 0$ implies that $\{x_n\}$ has a subsequence converging to an element in $F(y)$, then y is said to be *generalized Tykhonov well-posed* with respect to φ;
(ii) If $F(y) = \{x^*\}$ (a singleton) and for any $x_n \in X$, $\varphi(x_n) \to 0$ implies that $x_n \to x^*$, then y is said to be *Tykhonov well-posed* with respect to φ.

Definition 2.1 is a unified approach to Tykhonov well-posedness, which can be applied to various nonlinear problems. For instance, for an optimization problem

$$\inf_{x \in A} f(x)$$

we may define $\varphi(x) = f(x) - \inf_{u \in A} f(u)$; for Nash equilibrium point problems, let

$$\varphi(x) = \sum_{i=1}^{n} [\sup_{u_i \in X_i} f_i(u_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})],$$

and for fixed point problems of set-valued maps, let

$\varphi(x) = d(x,f(x))$.

(The notations above will be described precisely in the sequel). Then we obtain the Tykhonov well-posedness for such problems.

The following definition is a unified approach to Hadamard well-posedness.

DEFINITION 2.2. Let $y \in Y$.

(i) If for any $y_n \to y$, any $x_n \in F(y_n)$, x_n must have a subsequence converging to an element in $F(y)$, then y is said to be *generalized Hadamard well-posed*;

(ii) If $F(y) = \{x^*\}$ (a singleton) and for any $y_n \to y$, any $x_n \in F(y_n)$, x_n must converge to x^*, then y is said to be *Hadamard well-posed.*

The following Theorem 2.1 shows the relation between Tykhonov well-posedness and Hadamard well-posedness.

THEOREM 2.1. *Let $y \in Y$ and suppose that there is $\varphi: X \to R_+$ such that $\varphi(x) = 0$ if and only if $x \in F(y)$. For any $x_n \in X$, if the following condition holds*:

$$\varphi(x_n) \to 0 \Rightarrow \exists y_n \in Y \text{ with } x_n \in F(y_n) \text{ such that } y_n \to y, \tag{1}$$

then

(a) *that $y \in Y$ is generalized Hadamard well-posed implies that $y \in Y$ is generalized Tykhonov well-posed with respect to φ;*

(b) *that $y \in Y$ is Hadamard well-posed implies that $y \in Y$ is Tykhonov well-posed with respect to φ.*

Proof. (a) Let (1) hold and y be generalized Hadamard well-posed. Let us show that y is generalized Tykhonov well-posed. Indeed, if $\varphi(x_n) \to 0$, then by (1), there is $y_n \in Y$ such that $x_n \in F(y_n)$ and $y_n \to y$. Thus x_n must have a subsequence convergent to an element in $F(y)$. Hence y is generalized Tykhonov well-posed.

The same argument can also be applied to prove (b). □

THEOREM 2.2. *Let X and Y be two Hausdorff topological spaces and $F: Y \to 2^X$ be a set-valued map. If $y \in Y$, then*

(a) *if F is upper semicontinuous at $y \in Y$ and $F(y)$ is compact, then y is generalized Hadamard well-posed,*

(b) *if F is upper semicontinuous at $y \in Y$ and $F(y) = \{x^*\}$, then y is Hadamard well-posed.*

Proof. (a) Equivalently, we prove that $\{x_n\}$ has a cluster point in $F(y)$. By way of contradiction, suppose that $\{x_n\}$ has no cluster point in $F(y)$. For each $u \in F(y)$, there exist an open neighborhood $O(u)$ of u in X and a positive integer $n(u)$ such that $x_n \notin O(u)$ for all $n \geqslant n(u)$. Since $F(y) \subset \cup_{u \in F(y)} O(u)$ and $F(y)$ is compact, there exist $u_1, u_2, \ldots, u_k \in F(y)$ such that $F(y) \subset \cup_{i=1}^{k} O(u_i)$. Now let n' be such that $n' \geqslant n(u_i)$ for $i = 1, 2, \ldots, k$, then for any $n \geqslant n'$, $x_n \notin O(u_i)$ for $i = 1, 2, \ldots, k$. Since F is upper semicontinuous at $y \in Y$, for open set $\cup_{i=1}^{k} O(u_i)$ with $\cup_{i=1}^{k} O(u_i) \supset F(y)$, there exists $n'' \geqslant n'$ such that $x_n \in F(y_n) \subset \cup_{i=1}^{k} O(u_i)$ for all $n \geqslant n''$. Now we have $x_{n''} \notin O(u_i)$ for $i = 1, 2, \ldots, k$ and $x_{n''} \in F(y_{n''}) \subset \cup_{i=1}^{k} O(u_i)$, a contradiction. Hence $\{x_n\}$ has a cluster point $\bar{x} \in F(y)$, that is, $\{x_n\}$ has a subsequence converging to $\bar{x} \in F(y)$.

(b) Also by way of contradiction, assume that $\{x_n\}$ does not converge to x^*. Then there exists an open set O with $x^* \in O$ such that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ with $x_{n_k} \notin O$. Obviously, any subsequence of $\{x_{n_k}\}$ does not converge to x^*. But, by (a), $\{x_{n_k}\}$ has a subsequence converging to x^*. This is a contradiction and completes the proof. □

3. Applications

In this section, we apply the unified approach in the previous section to investigate well-posedness for optimization problems, Nash equilibrium problems and fixed point problems etc.

First we consider well-posed Nash equilibrium problems. Let $N = \{1, \ldots, n\}$ be the set of players. For any $i \in N$, denote $\hat{i} = N \setminus i$. For any $i \in N$, X_i, a nonempty compact subset of a Hausdorff topological space E_i, is the strategy set of player i and $f_i : X = \prod_{i=1}^{n} X_i \to R$ which is continuous on X is the payoff function of player i.

Let C be the collection of all $f = (f_1, \ldots, f_n)$ satisfying the above conditions. For any $f = (f_1, \ldots, f_n) \in C$, $g = (g_1, \ldots, g_n) \in C$, define

$$\rho(f, g) = \sup_{x \in X} \sum_{i=1}^{n} |f_i(x) - g_i(x)|.$$

Then (C, ρ) becomes a metric space.

Let $M = \{f = (f_1, \ldots, f_n) \in C \mid$ there is Nash equilibrium $x^* \in X$ for f, i.e., for any $i \in N$, $f_i(x_i^*, x_{\hat{i}}^*) = \sup_{u_i \in X_i} f_i(u_i, x_{\hat{i}}^*)\}$. For any $f \in M$, denote by $F(f)$ the set of all Nash equilibria of f. Then F defines a set-valued map from M into X. By Theorem 3.3 of [11], F is an usco map. Note that in [11], in order to guarantee the existence of Nash equilibria, in addition to the continuity conditions, it is required that f satisfies some convexity conditions which is not necessary for the proof of continuity of F. The definition of M guarantees the existence of Nash equilibria.

Hence every $f \in M$ is generalized Hadamard well-posed and if furthermore, $F(f) = \{x^*\}$, then f is Hadamard well-posed.

Let $\varphi(x) = \sum_{i=1}^{n}[\sup_{u_i \in X_i} f_i(u_i, x_{\hat{i}}) - f_i(x_i, x_{\hat{i}})]$, then for any $x \in X$, $\varphi(x) \geqslant 0$ and for any $u \in F(f)$, $\varphi(u) = 0$. For any $x^m \in X$ with $\varepsilon_i^m := \sup_{u_i \in X_i} f_i(u_i, x_{\hat{i}}^m) - f_i(x_i^m, x_{\hat{i}}^m) \to 0$, i.e., $\varphi(x^m) \to 0$, $i \in N$, define

$$f_i^m(x) = \begin{cases} f_i(x) + \varepsilon_i^m, & f(x) \leqslant a_i^m - 2\varepsilon_i^m, \\ a_i^m - \varepsilon_i^m, & a_i^m - 2\varepsilon_i^m < f_i(x) \leqslant a_i^m, \\ f_i(x) - \varepsilon_i^m, & a_i^m < f_i(x) \leqslant A_i, \end{cases}$$

where $a_i^m = \max_{u_i \in X_i} f_i(u_i, x_{\hat{i}}^m)$ and $A_i = \max_{x \in X} f_i(x)$. Then, it can be routinely verified that $f^m \in M$, $x^m \in F(f^m)$ and

$$\rho(f^m, f) = \sup_{x \in X} \sum_{i=1}^{n} |f_i^m(x) - f_i(x)| \leqslant \sum_{i=1}^{n} \varepsilon_i^m \to 0.$$

Hence, by Theorem 2.1, every $f \in M$ must be generalized Tykhonov well-posed and if $F(f)$ is a singleton then f is Tykhonov well-posed.

As a special case, when $n = 1$, we obtain well-posedness for optimization problems. Let X be a compact Hausdorff topological space and let $C(X)$ be the space of all continuous real-valued functions, endowed with the uniform norm. Then, every $f \in C(X)$ is generalized Hadamard well-posed and if, furthermore $F(f) = \{x^*\}$ then f is Hadamard well-posed.

Let $\varphi(x) = f(x) - \inf_{u \in X} f(u)$. Then the similar argument implies that every $f \in C(X)$ must be generalized Tykhonov well-posed with respect to φ and if $F(f)$ is a singleton, then f is Tykhonov well-posed with respect to φ.

It is also should be noted that the notions of Tykhonov well-posedness defined above for Nash equilibria and optimization problems are the same as the usual sense.

Finally we consider well-posed fixed point problems. Let X be a compact metric space and $K(X)$ be the set of all nonempty compact subsets of X. For any $A, B \in K(X)$, $h(A, B)$ denotes the Hausdorff distance between A and B.

Let $C = \{f : X \to K(X) \mid f$ is upper semicontinuous on $X\}$ and for any $f, g \in C$, define

$$\rho(f, g) = \sup_{x \in X} h(f(x), g(x)).$$

Let $Y = C \times K(X)$ and for any $y = (f, A) \in Y$, $y' = (f', A') \in Y$ define

$$D(y, y') = \rho(f, f') + h(A, A'),$$

then (Y, D) is a metric space.

Let $M = \{y = (f, A) \in Y \mid \exists x \in A, x \in f(x)\}$. For any $y \in M$, define $F(y) = \{x \in A \mid x \in f(x)\}$, i.e., $F(y)$ is the set of all fixed points of f in A. Then F defines a set-valued map from M into X and F can be proved to be an usco map by applying the argument in the proof of Lemmas 5 and 6 of [12].

By Theorem 2.2, every $y \in M$ is generalized Hadamard well-posed and if, furthermore, $F(y)$ is a singleton then y is Hadamard well-posed.

Now denote $\varphi(x) = d(x, f(x)) = \inf_{u \in f(x)} d(x, u)$, where $d(x, u)$ denotes the distance between x and u in X. For any $x \in X$ we have $\varphi(x) \geqslant 0$ and for any $u \in F(y)$, $\varphi(u) = 0$. For any $x_n \in X$ with $d(x_n, f(x_n)) \to 0$, denote $\delta_n = d(x_n, f(x_n))$. Define $f_n: X \to K(X)$ as follows: for any $x \in X$,

$$f_n(x) = \{u \in X \mid d(u, f(x)) \leqslant \delta_n\},$$

and we denote $A_n = A, y_n = (f_n, A_n)$.

It can be checked that $f_n \in C$, $y_n = (f_n, A_n) \to (f, A)$ and $x_n \in f_n(x_n)$, i.e., $y_n \in M$, $x_n \in F(y_n)$. By Theorem 2.1, every $y \in M$ must be generalized Tykhonov well-posed with respect to φ and if, furthermore, $F(y)$ is a singleton, then y must be Tykhonov well-posed with respect to φ.

4. Well-Posedness for OPVIC

Following [1], let X be a topological space, E be a Banach space with dual space E^* and K be a nonempty convex closed subset of E. Let E' be the collection of all affine functionals defined on E satisfying that there exists $A \in E^*$ and $\lambda \in R$ such that $\langle \varphi, w \rangle = \langle A, w \rangle + \lambda$ for any $w \in E$.

Given a function $f: X \times E \to R$, an optimization problem with variational inequality constraints, denoted by OPVIC, can be stated as follows:

$$(\text{OPVIC}) \begin{cases} \min f(x, u) \\ \text{subject to } (x, u) \in X \times E \text{ and } u \in T(x), \end{cases} \tag{2}$$

where $T(x)$ is solution set of parametric variational inequality $(VI)(x)$ defined by the pair $(\varphi(x, \cdot), K)$, $\varphi(x, \cdot)$ being an operator from E to E', i.e., $u \in T(x)$ if and only if $u \in K$ and satisfies the inequalities:

$$\langle \varphi(x, u), u - v \rangle \leqslant 0, \ \forall v \in K.$$

REMARK. In [1], in the constraints the parametric variational inequality $(VI)(x)$ is defined by the pair $(A(x, \cdot), K)$ where $A(x, \cdot)$ is an operator from E to E^*. Hence the OPVICs here include that in [1] as special cases.

In [1], Lignola and Morgan introduced Tykhonov type of well-posedness for OPVICs, namely, generalized well-posedness and strong well-posedness. In this section, we present an alternative approach to the study of well-posedness for OPVICs. In principle, we first investigate Hadamard type of well-posedness of OPVIC and we determine classes of problems which guarantee some Hadamard well-posedness. Secondly, we deduce the relations between Hadamard type of well-posedness and Tykhonov type of well-posedness defined in [1]. Our results include the corresponding results in [1] as special cases.

DEFINITION 4.1. Let X and Y be two topological spaces and $F: Y \to 2^X$ be a set-valued map. F is said to be a sequentially closed map if its graph $\mathrm{Gr}F = \{(y,x) \in Y \times X \mid x \in F(y)\}$ is sequentially closed, i.e., for any sequence $\{(y_n, x_n)\} \in \mathrm{Gr}F$, if $(y_n, x_n) \to (y_0, x_0)$, then $(y_0, x_0) \in \mathrm{Gr}F$.

In the following, we recall some notions defined in [1].

DEFINITION 4.2. Let $x \in X$ and $\{x_n\}$ be a sequence converging to x. A sequence $\{u_n\}$ is said to be an *approximating sequence* for the problem $(VI)(x)$ (with respect to $\{x_n\}$) if $u_n \in K$ for any n and there exists a sequence of positive numbers $\{\varepsilon_n\}$ converging to zero such that

$$\langle \varphi(x_n, u_n), u_n - v \rangle \leqslant \varepsilon_n, \ \forall v \in K.$$

Denote $(VI) = \{(VI)(x), x \in X\}$.

DEFINITION 4.3. The family (VI) is *parametrically strongly well-posed* if:

(i) there exists a unique solution $\bar{u_x}$ to $(VI)(x)$, for all $x \in X$;
(ii) for all $x \in X$ and for all $\{x_n\}$ converging to x, every approximating sequence for the problem $(VI)(x)$ (with respect to $\{x_n\}$) strongly converges to $\bar{u_x}$.

Let X be a sequentially compact topological space. Let C be the collection of all maps $A : X \times E \to E^*$ and let Γ be the collection of all maps $\varphi : X \times E \to E'$ such that for any $(x,u) \in X \times E$, there is $A \in C$ and $\lambda \in R$ such that $\langle \varphi(x,u), w \rangle = \langle A(x,u), w \rangle + \lambda, \ \forall w \in E$. For any $\varphi_1 = (A_1, \lambda_1)$, $\varphi_2 = (A_2, \lambda_2) \in \Gamma$, define

$$d(\varphi_1, \varphi_2) = \begin{cases} |\lambda_1 - \lambda_2|, & A_1 = A_2 \\ 1 + |\lambda_1 - \lambda_2|, & A_1 \neq A_2 \end{cases}$$

then it can be routinely checked that (Γ, d) is a metric space.

Let M_1 be the collection of all (f, φ) such that

(i) f is lower semicontinuous on $X \times (E, s)$, where (E, s) denotes the space E with the strong topology;
(ii) The family (VI) is parametrically strongly well-posed; and
(iii) $\sup\limits_{(x,u) \in X \times E} |f(x,u)| < +\infty$.

Then for each $p = (f, \varphi) \in M_1$, we obtain an OPVIC defined by (2). For any $p_1, p_2 \in M_1$, define

$$\rho(p_1, p_2) = \sup_{(x,u)\in X\times E} |f_1(x,u) - f_2(x,u)| + d(\varphi_1, \varphi_2),$$

Clearly, (M_1, ρ) is a metric space. Let $M = \{p \in M_1 : p$ admits at least one solution$\}$. For each $p \in M$, denote by $F(p)$ the solution set of p. Then F defines a set-valued map from M into $X \times K$.

DEFINITION 4.4. Let $p \in M$.

(i) p is said to be *generalized Hadamard well-posed* with respect to M if for any sequence $\{p_n\} \subset M$ with $p_n \to p$ and any sequence $\{(x_n, u_n)\} \subset X \times E$ with $(x_n, u_n) \in F(p_n)$, (x_n, u_n) has a subsequence converging to an element in $F(p)$.

(ii) p is said to be *Hadamard well-posed* with respect to M if p possesses a unique solution (x^*, u^*) and, for any sequence $\{p_n\} \subset M$ with $p_n \to p$ and any sequence $\{(x_n, u_n)\} \subset X \times E$ with $(x_n, u_n) \in F(p_n)$, (x_n, u_n) must converges to (x^*, u^*).

LEMMA 4.1. *$F : M \to 2^{X\times K}$ is a sequentially closed set-valued map, i.e., $\mathrm{Gr}F = \{(p, (x,u)) \in M \times X \times K \mid (x,u) \in F(p)\}$ is sequentially closed in $M \times X \times K$.*

Proof. Let $\{(p_n, (x_n, u_n))\}$ be any sequence in $\mathrm{Gr}F$ with $(p_n, (x_n, u_n)) \to (p_0, (x_0, u_0)) \in M \times X \times K$. We will show that $(p_0, (x_0, u_0)) \in \mathrm{Gr}F$, i.e., $(x_0, u_0) \in F(p_0)$. Denote $p_n = (f_n, \varphi_n)$, $p_0 = (f_0, \varphi_0)$. First, since f_0 is lower semicontinuous on $X \times (E, s)$, for any $\varepsilon > 0$, there exists n_0 such that $f_0(x_0, u_0) \leqslant f_0(x_n, u_n) + \varepsilon$ for all $n \geqslant n_0$. Since $p_n \to p_0$, there exists n_1 such that $f_0(x,u) - \varepsilon \leqslant f_n(x,u) \leqslant f_0(x,u) + \varepsilon$ for all $(x,u) \in X \times E$ whenever $n \geqslant n_1$. Let n_2 be such that $n_2 \geqslant n_1$ and $n_2 \geqslant n_0$, then

$$f_0(x_0, u_0) \leqslant f_0(x_n, u_n) + \varepsilon \leqslant f_n(x_n, u_n) + 2\varepsilon, \ \forall n \geqslant n_2. \tag{3}$$

Since $(x_n, u_n) \in F(p_n)$, then $u_n \in T_n(x_n)$ and

$$f_n(x_n, u_n) \leqslant f_n(x,u), \quad \forall (x,u) \in X \times E. \tag{4}$$

Combining (3) and (4), we have

$$f_0(x_0, u_0) \leqslant f_n(x,u) + 2\varepsilon \leqslant f_0(x,u) + 3\varepsilon, \ \forall (x,u) \in X \times E.$$

Now let ε go to zero, then we have $f_0(x_0, u_0) \leqslant f_0(x,u), \ \forall (x,u) \in X \times E$.

Since $\varphi_n \to \varphi_0$, by the definition of d, there must be $A_n = A_0 = A$ and $d(\varphi_n, \varphi_0) = |\lambda_n - \lambda_0| \to 0$. Since $u_n \in T_n(x_n)$,

$$\langle \varphi_n(x_n, u_n), u_n - v\rangle = \langle A(x_n, u_n), u_n - v\rangle + \lambda_n \leqslant 0, \ \forall v \in K.$$

Hence

$$\langle \varphi_0(x_n, u_n), u_n - v\rangle = \langle A(x_n, u_n), u_n - v\rangle + \lambda_0 \leqslant \langle A(x_n, u_n), u_n - v\rangle + \lambda_n + |\lambda_0 - \lambda_n| \leqslant |\lambda_0 - \lambda_n| \to 0, \quad \forall v \in K$$

i.e., $\{u_n\}$ is an approximating sequence for $(VI)(x_0)$ (with respect to $\{x_n\}$). By the parametrically strong well-posedness of (VI), $(VI)(x_0)$ has a unique solution $\bar{u}_{x_0}$ towards which $\{u_n\}$ converges. It follows that $u_0 = \bar{u}_{x_0}$ since $u_n \to u_0$, and thus $(x_0, u_0) \in F(p_0)$.

The proof is complete. □

THEOREM 4.1. *Every $p \in M$ is generalized Hadamard well-posed and, furthermore, if $F(p)$ is a singleton then p is Hadamard well-posed.*

Proof. Let $\{p_n\} \subset M$ be any sequence with $p_n \to p$ and $(x_n, u_n) \in X \times E$ with $(x_n, u_n) \in F(p_n)$. Since X is sequentially compact, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ and we may assume that $x_{n_k} \to x_0 \in X$. Since $(x_n, u_n) \in F(p_n)$, we have $\langle \varphi(x_{n_k}, u_{n_k}), u_{n_k} - v\rangle \leqslant 0$ for all $v \in K$, which implies that $\{u_{n_k}\}$ is an approximating sequence for $(VI)(x_0)$ (with respect to $\{x_{n_k}\}$). By the parametrical well-posedness of $(VI)(x_0)$, there exists a unique solution u_0 to $(VI)(x_0)$ towards which $\{u_{n_k}\})$ strongly converges and thus $(p_{n_k}, (x_{n_k}, u_{n_k})) \to (p, (x_0, u_0))$. By Lemma 4.1, $(p, (x_0, u_0)) \in \mathrm{Gr}F$, i.e., $(x_0, u_0) \in F(p)$. Hence p is generalized Hadamard well-posed.

The similar argument as stated in the proof of Theorem 2.2 (b) can be applied to prove the second assertion. Now the proof is complete. □

DEFINITION 4.5. A sequence $\{(x_n, u_n)\}$ is an *approximating sequence* for $p \in M$ if:

(i) $\liminf_{n\to\infty} f(x_n, u_n) \leqslant \inf_{\substack{(x,u)\in X\times E \\ u\in T(x)}} f(x, u)$; and

(ii) $\langle \varphi(x_n, u_n), u_n - v\rangle \leqslant \varepsilon_n$, $\forall v \in K$, where $\varepsilon_n \geqslant 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

DEFINITION 4.6. An OPVIC is generalized well-posed if:

(i) $(VI)(x)$ has a unique solution for every $x \in X$;
(ii) OPVIC has at least a solution; and
(iii) any approximating sequence $\{(x_n, u_n)\}$ for OPVIC has a subsequence convergent in $X \times (E, s)$ to a solution of OPVIC.

DEFINITION 4.7. An OPVIC is strongly well-posed if:

(i) OPVIC has a unique solution (x^*, u^*);
(ii) any approximating sequence $\{(x_n, u_n)\}$ for OPVIC converges to (x^*, u^*) in $X \times (E, s)$.

THEOREM 4.2. *In space M, the following assertions hold*:

(a) *generalized Hadamard well-posedness implies generalized well-posedness*;
(b) *Hadamard well-posedness implies strong well-posedness*.

Proof. Let $p = (f, \varphi) \in M$ and let $\{(x_n, u_n)\}$ be an approximating sequence for p, i.e.,

(i) $\liminf_{n\to\infty} f(x_n, u_n) \leqslant \inf_{\substack{(x,u)\in X\times E \\ u\in T(x)}} f(x, u)$; and

(ii) $\langle \varphi(x_n, u_n), u_n - v\rangle \leqslant \varepsilon_n$, $\forall v \in K$, where $\varepsilon_n \geqslant 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$.

Denote $\delta_n = f(x_n, u_n)$. We construct a sequence $\{p_n = (f_n, \varphi_n)\}$ as follows: for each $(x, u) \in X \times E$,

$$f_n(x, u) = \begin{cases} f(x, u) - \delta_n, & f(x, u) \geqslant a + 2\delta_n, \\ a + \delta_n, & a \leqslant f(x, u) < a + 2\delta_n. \end{cases}$$

where $a = \inf_{\substack{(x,u)\in X\times E \\ u\in T(x)}} f(x, u)$, and

$$\langle \varphi_n(x, u), w\rangle = \langle \varphi(x, u), w\rangle - \varepsilon_n, \quad \forall w \in E$$

It can be routinely checked that

(i) f_n is lower semicontinuous on $X \times (E, s)$;
(ii) $\langle \varphi_n(x_n, u_n), u_n - v\rangle \leqslant 0, \forall v \in K$, i.e., $u_n \in T_n(x_n)$; and
(iii) $f_n(x_n, u_n) = \inf_{\substack{(x,u)\in X\times E \\ u\in T_n(x)}} f_n(x, u) = a + \delta_n$

Hence $p_n \in M$ and $(x_n, u_n) \in F(p_n)$. Now we can deduce the following two statements:

(a) if p is generalized Hadamard well-posed, then $\{(x_n, u_n)\}$ has a subsequence converging to an element in $F(p)$. Thus p is generalized well-posed.
(b) if p is Hadamard well-posed, then p has a unique solution towards which $\{(x_n, u_n)\}$ converges. Hence p is strongly well-posed.

The proof is complete. □

Now we state the following corollary (see [1], Theorems 3.4 and 3.5).

COROLLARY 4.1. *Every $p \in M$ is generalized well-posed. Furthermore, if $F(p)$ is a singleton then p is strongly well-posed.*

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